Derivations for High School Physics

Justin Ely

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Preface

Oftentimes, equations are given to students on various equation sheets on physics tests. Although knowing the equations is important, it is even more important to know the derivations of the equations as it provides a deeper understanding of the physical concepts behind the equations and builds physical intuition required to solve problems. Using the mathematical flexibility gained through proving formulas, students will better be able to apply their knowledge to solve more complex physics problems.

These notes attempt to derive as many formulae and concepts as possible that pertain to high school physics, from AP courses to Physics Olympiads. Equations featured in AP Physics 1 and 2 are derived without calculus (as much as possible), while calculus is employed when necessary to derive more difficult expressions. Vectors are represented using boldface (**v**) in this text, and, a star (★) is used to mark the conclusion of each concept/derivation.

The sections required for various tests are marked with the following colors:

- ▶ AP Physics 1
- ▶ AP Physics 2
- ▶ AP Physics C
- ▶ F=ma Exam
- ▶ USAPhO Exam

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Chapter 1

Thermodynamics

1.1 Statistical Thermodynamics

1.1.1 ▶ The Maxwell-Boltzmann Distribution

An ideal gas can be represented by a Gaussian distribution of the velocity of the singular particles in the gas, given by the following distribution:

$$f(v) = \frac{4}{\sqrt{\pi}} \left[\frac{m}{2k_BT}\right]^{3/2} v^2 e^{\frac{-mv^2}{2k_BT}} \tag{1.1}$$

The function is known as the **Maxwell-Boltzmann Distribution** and although it is a bit messy, it's important to recognize what it means.

The function is a **Probability Distribution Function (PDF)**, which is a special type of function that describes the way in which a random system is arranged. If we consider an arbitrary PDF $\rho(x)$ where x is some state of the system, ρ has no real meaning at any value of x. However, the area under ρ (integral) represents the probability of the sytem taking on a state that lies between the bounds of integration. Thus, any PDF $\rho(x)$ has the property

$$\int_{-\infty}^{\infty} \rho(x)dx = 1 \tag{1.2}$$

since the system must always be occupying one of its possible states. Similarly, the probability of an event occurring in the interval $(x, x + dx)$ is $\rho(x)dx$. For a container of gas, the particles are all evenly distributed, meaning that the probability of a gas particles having a velocity in the range $(v, v + dv)$ equals the fraction of all the particles with that velocity. Or,

$$\frac{dN}{N} = f(v)dv \tag{1.3}$$

where dN is the number of gas particles with a speed in the range $(v, v + dv)$. Recall that the expected value (or mean) of a system is defined as

$$\langle x\rangle = \sum_i x_i p(x_i)$$

where $\langle x\rangle$ is the average of x, x_i is an arbitrary state, and $p(x_i)$ is the probability of x_i occurring. It follows that for continuous distributions

$$\langle x\rangle = \int_{-\infty}^{\infty} x\rho(x)dx \tag{1.4}$$

since x is a state, $\rho(x)dx$ is the probability, and we are again summing over all possible states. Similarly, the mean of any function of x is

$$\langle g(x)\rangle = \int_{-\infty}^{\infty} g(x)\rho(x)dx. \tag{1.5}$$

The reason why we don't change $\rho(x)$ is because the fundamental randomization process remains unchanged, and we are simply rescaling the quantity. For example, if we wanted to find the average of the square of a die roll, we would multiply the outcomes $(1, 4, 9, 16, 25, 36)$ by the probability $1/6$ and add them up to find the mean, which is the same probability used to calculate the mean value of the die normally. This all makes sense, because we aren't changing anything about the die roll, just the output value of each state. At any rate, these properties will be important when we examine macro-scale properties of systems of small particles.

Going back to the Maxwell-Boltzmann distribution, we can break it into two parts:

$$f(v) = \frac{4}{\sqrt{\pi}}\left[\frac{m}{2k_BT}\right]^{3/2}$$

$$\cdot\, v^2 e^{\frac{-mv^2}{2k_BT}}$$

The bottom part of the distribution relates to the actual distribution of the speeds of the particles in the gas[1] and is derived using statistical physics and experiment. The top part is a normalization factor, which gives the function the property from eq. 2.2. The Maxwell-Boltzmann distribution is special, though, since it's normalized in such a way that the area under the curve from 0 to ∞ is one, instead of the area along the entire x axis. To see why, consider the following graph of $f(v)$:

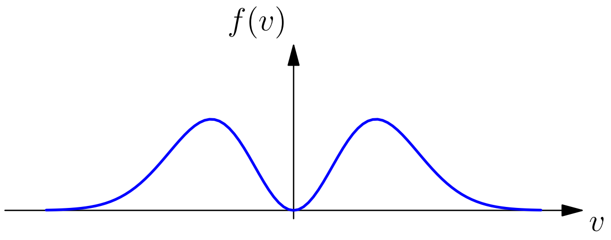

Fig. 1.1: Maxwell-Boltzmann Distribution

Since objects cannot have negative speed, all of the particles in a gas must lie on the right side of the graph, meaning

$$\int_0^\infty f(v)dv = 1. \tag{1.6}$$

This somewhat unusual normalization will be important in 1.1.2 when we integrate over all particles ★

1.1.2 ▶ Root Mean Square Velocity

The root mean square (RMS) velocity is defined as the square root of the average of the squares of the velocities of the particles in a system. Mathematically (and possibly much more simply),

$$v_{RMS} = \sqrt{\langle v^2 \rangle}. \tag{1.7}$$

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[1] i.e. it's the part that gives the graph its shape

RMS quantities appear a lot in statistical physics, and they are used preferential to normal averages for a couple reasons:

1: The average value of an oscillating function may be zero, independent of the amplitude of the oscillations. However, it's hardly intuitive for a wildly-oscillating function to be described the same way as a barely-changing function. The RMS value fixes this discrepancy since it squares all values before taking an average, providing a more accurate representation of a quantity.

2: As we see in 1.1.3, since kinetic energy is defined with the square of the velocity, average velocity is useless if we want to calculate the average kinetic energy. Thus, v_{RMS} is used instead of the normal average.

In any case, since we know the PDF for particles in a gas, we can use the definitions listed in 1.1.1 to find a general expression. First, we can find the quantity $\langle v^2\rangle$ by (notice the bounds of integration from eq. 2.6)

$$\langle v^2\rangle = \int_0^\infty v^2 f(v)dv \tag{1.8}$$

where $f(v)$ is the Maxwell-Boltzmann distribution. Plugging in $f(v)$ and rearranging yields

$$\frac{2}{\sqrt{\pi}}a^{3/2}\int_0^\infty v^4 e^{-av^2}dv.$$

Integrating by parts with $u = v^3, du = 3v^2dv$ and $dw = ve^{-av^2}dv, w = -\frac{1}{2a}e^{-av^2}$, we have

$$\frac{2}{\sqrt{\pi}}a^{3/2}\left[-\frac{v^3}{2a}e^{-av^2}\bigg|_{-\infty}^{\infty} + \frac{3}{2a}\int_0^\infty v^2 e^{-av^2}dv\right] \tag{1.9}$$

The first term in the brackets is zero since the exponential decays to zero at infinity. To find the integral on the right, we could use integration by parts again, but instead we can use Feynman's trick for a quicker solution. By properties of the Gaussian integral [2],

$$\int_0^\infty e^{-av^2}dv = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

Comparing this property with the integral in eq. 2.8, we can observe the following clever relation

$$\int_0^\infty v^2 e^{-av^2}dv = -\frac{d}{da}\int_0^\infty e^{-av^2}dv$$

where we take a derivative with respect to a, since the integral is a function of a [3]. Thus, the expression in eq. 2.8 becomes

$$\begin{aligned}\langle v^2\rangle &= \frac{4}{\sqrt{\pi}}a^{3/2}\left[\frac{3}{2a}\left(-\frac{d}{da}\frac{1}{2}\sqrt{\frac{\pi}{a}}\right)\right]\\ &= \frac{4}{\sqrt{\pi}}a^{3/2}\left[\frac{3}{2a}\left(\frac{1}{4}\frac{\sqrt{\pi}}{a^{3/2}}\right)\right]\\ &= \frac{3}{2a}. \end{aligned}\tag{1.10}$$

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[2] I won't prove it here, but the derivation is very interesting and you should definitely check it out!

[3] v is a dummy variable, and is integrated out

The RMS speed is the square root of this value, or

$$\begin{aligned} v_{rms} &= \sqrt{\langle v^2 \rangle} \\ &= \sqrt{\frac{3}{2a}} \\ &= \sqrt{\frac{3k_BT}{m}} \end{aligned} \tag{1.11}$$

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1.1.3 ▶ ▶ Internal Energy

The total internal energy of a thermodynamic system is defined as the total energy contained within it. For an ideal gas, since there are no internal potentials, this energy is simply the total kinetic energy of the gas particles, or

$$U_{int} = N\langle K \rangle \tag{1.12}$$

where N is the number of particles and $\langle K \rangle$ is the number of gas particles. Using the mechanical definition of kinetic energy,

$$\langle K \rangle = \frac{1}{2}m\langle v^2 \rangle = \frac{1}{2}mv_{rms}^2$$

Hence, the total internal energy of a monatomic gas is

$$U_{int} = \frac{3}{2}K_BT. \tag{1.13}$$

Alternatively, if we define n to be the number of moles of gas constant and the constant R to be Avogadro's number times k_B, we have

$$U_{int} = \frac{3}{2}(nN_A)k_BT = \frac{3}{2}nRT \tag{1.14}$$

Notice how this result is specifically for *monatomic* gas, and you can think of the equation as the sum of the average kinetic energy in each degree of freedom (x,y,z axis), which gives rise to the factor of 3/2.

$$K_i = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2$$

For diatomic gasses, there are two[4] additional rotational degrees of freedom, namely

$$K_i = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2 + \frac{1}{2}I_x\boldsymbol{\omega}_x^2 + \frac{1}{2}I_y\boldsymbol{\omega}_y^2$$

Logically, the total internal energy should be

$$U_{int} = \frac{5}{2}Nk_BT \tag{1.15}$$

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[4] Two, not three, since there is symmetry about two axes. For objects with less symmetry, there will be three rotational axes (Try it! Try to find more than two unique rotational axes for a pencil, then try the same with a smartphone!).

1.1.4 ▶ Ideal Gas Law

The famous Ideal Gas Law $PV = nRT$ was originally derived by experiment as an amalgamation of other emperical laws, namely Boyle's Law, Charles' Law, Gay-Lussac's Law, and Avogadro's Law. However, using the kinetic theory of gasses, it's possible to derive the law from first principles.

We start by considering a cube with side length L filled with a monatomic ideal gas. Since the gas exerts equal pressure in all directions, if we can relate the pressure of particles bouncing off of a single wall to the box's volume and temperature, we simply get the ideal gas law. For this derivation, we look at the particles bouncing off of the wall in the positive x axis (shown in blue). In a small time Δt, some particles traveling in the positive x direction collide with the blue surface, and those particles take up the volume defined by $\Delta l = \overline{v_x}\Delta t$ where $\overline{v_x}$ is the average x component of the velocities of the gas particles.

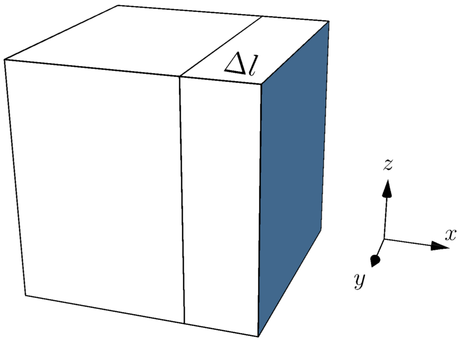

Fig. 1.2: Box with gas particles randomly moving inside. In a certain time Δt, some particles inside the section Δl hit the blue wall.

In this section of the cube, the number of particles is the number density of particles in the entire box multiplied by the volume of the section, since the particles are all evenly distributed. That number is

$$\frac{N}{V}\Delta l A = \frac{N A v_x \Delta t}{V}.$$

Of these particles that move on the x axis, half of them move towards the blue surface and half away. However, only those that move toward the surface exert any pressure, bringing our total considered particles to

$$\frac{N A v_x \Delta t}{2V}. \tag{1.16}$$

Since the gas is ideal, it collides elastically withe the walls of the container, imparting a force of $F = 2mv_x/\Delta t$ each. Thus, using eq. 2.16, the total force imparted on the blue side is

$$\frac{N A m v_x^2}{V}. \tag{1.17}$$

Since $P = F/A$, the average pressure on the blue side is

$$\langle P\rangle = \frac{Nm\langle v_x^2\rangle}{V}. \tag{1.18}$$

Since the particles are moving in all directions randomly, we can write the relation $v_{rms}^2 = \langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle = 3\langle v_x^2 \rangle$, which implies that $\langle v_x^2 \rangle = \frac{k_B T}{m}$. Substituting everything in, we have[5]

$$P = \frac{N k_B T}{V}$$

$$PV = N k_B T \tag{1.19}$$

$$PV = nRT \tag{1.20}$$

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[5] I removed the angle brackets around P here since the P in $PV = nRT$ represents the average pressure of the gas in a certain state of the gas. More specifically, the P in $PV = nRT$ represents the "average" over all of the microstates that a given macrostate encompasses.

1.2 Macroscopic Thermodynamics

1.2.1 ▶ ▶ Work Done on an Ideal Gas

Finding the work done on an ideal gas is important, since we can combine it with the first law of thermodynamics to derive some interesting relations (as seen later). To start, consider cylinder of gas of volume V topped with a movable piston of area A. If we apply a force F to the piston and depress it by a *distance* Δx, the work done by this force is simply

$$W = F\Delta x.$$

Dividing and multiplying by A yields

$$W = \frac{F}{A}A\Delta x$$

$$W = -P\Delta V. \tag{1.21}$$

Notice here that $A\Delta x = -\Delta V$. Since the cylinder is compressed, ΔV is negative. Thus, we need to add a negative sign to make things work out. This negative sign makes sense, since we'd expect that compressing a gas would do positive work. Why? Similar to a spring, when you compress the gas you work against a force, you're increasing the energy of the system. At a molecular level, you are simply increasing the momentum of the particles in the box as they collide with the surface of the piston! (see 1.1.4) ★

1.2.2 ▶ ▶ Heat Capacity of Gasses and Meyer's Relation

The specific heat capacity C of a substance is defined as the ratio of the amount of heat it takes to raise the temperature of a substance per mole of substance. In other words,

$$Q = nC\Delta T \tag{1.22}$$

Solids and liquids have relatively constant heat capacities over most temperature intervals, owing to their low thermal expansion. Gasses, on the other hand, have variable heat capacities, primarily because of their ability to change volume. Thus, it's helpful to define a few heat capacities for gasses, and we can even derive some useful relations between them.

First is the specific heat capacity at constant volume or C_v. Since the volume of the gas is constant, no work can be done on it and thus it can only change energy via heat transfer. Plugging in appropriate expressions into the First Law of Thermodynamics, we have (for a monatomic gas)

$$\Delta U = Q$$

$$\frac{3}{2}nR\Delta T = nC_v\Delta T$$

$$C_v = \frac{3}{2}R \tag{1.23}$$

The other notable heat capacity is the heat capacity at constant pressure. Writing out the First Law of Thermodynamics for a constant pressure pressure yields

$$\begin{aligned} \Delta U &= W + Q \\ \frac{3}{2}nR\Delta T &= -P\Delta V + nC_p\Delta T. \end{aligned} \tag{1.24}$$

Since pressure is constant, we have from the Ideal Gas Law that

$$\Delta V = \frac{nR\Delta T}{P} \tag{1.25}$$

Substituting eq. 2.25 into 2.24 and simplifying gives us

$$\begin{aligned} \frac{3}{2}nR\Delta T &= -nR\Delta T + nC_p\Delta T \\ C_p &= \frac{5}{2}R \\ C_p &= C_v + R. \end{aligned} \tag{1.26}$$

Equation 2.26 is known as Mayer's Relation and provides a convenient relation between the heat capacity of a gas at constant volume and pressure. An important thing to see is that it doesn't say anything about the values of C_v or C_p, just their relation. In fact, for diatomic gasses, the values for C_v and C_p are $5R/2$ and $7R/2$ respectively, which is a consequence of the phenomenon discussed towards the end of 1.1.3 ★

1.2.3 ▶ The Adiabatic Curve

On a P/V diagram, isobaric processes are a horizontal line, isochoric processes are a vertical line, and isothermal processes are a $1/x$ curve. These are all pretty intuitive, and can be derived in a few steps. Less intuitive is the curve for an adiabatic process, which we will derive below. By definition of an adiabatic process, any changes in internal energy of a system must arise only from work done on the system, or

$$dU = \frac{3}{2}nRdT = \delta W \tag{1.27}$$

where δ represents the a small change in a path dependent quantity. Using 1.2.1 and the Ideal Gas Law, we have

$$\begin{aligned} \frac{3}{2}nRdT &= -PdV \\ nC_vdT &= -nR\frac{TdV}{V} \\ \frac{C_v}{R}\int\frac{dT}{T} &= -\int\frac{dV}{V} \\ \frac{C_v}{R}\ln T &= -\ln V + C \end{aligned} \tag{1.28}$$

where C is some constant. Rewriting R in terms of C_p and C_v as well as defining the ratio $\gamma = C_p/C_v$, we have

$$\begin{aligned}
\ln T &= \frac{C_v - C_p}{C_v} \ln V + C \\
\ln T &= (1-\gamma)\ln V + C \\
TV^{\gamma-1} &= C \qquad (1.29)
\end{aligned}$$

which itself is a perfectly good relation between quantities for an adiabatic process. However, since we're graphing things on a P/V diagram, it's nice to have a relation between P and V. Substituting the temperature using the Ideal Gas Law,

$$\begin{aligned}
\frac{PV}{nR}V^{\gamma-1} &= C \\
PV^{\gamma} &= const. \qquad (1.30)
\end{aligned}$$

Since $\gamma > 1$, this curve looks similar to the curve for isothermal processes, but is a bit steeper ★

1.2.4 ▶ Carnot's Theorem

Carnot's theorem is a direct result of the second law of thermodynamics, and is an important result for engine design. It states that the most efficient heat engine that runs between two heat reservoirs is a perfectly reversible engine. The corollary of this statement is that all reversible engines running between two reservoirs of different temperature have the same efficiency.

We start the proof by considering a reversible heat engine R with efficiency η_R and an arbitrary heat engine E with efficiency η_E operating between a hot and cold reservoir.

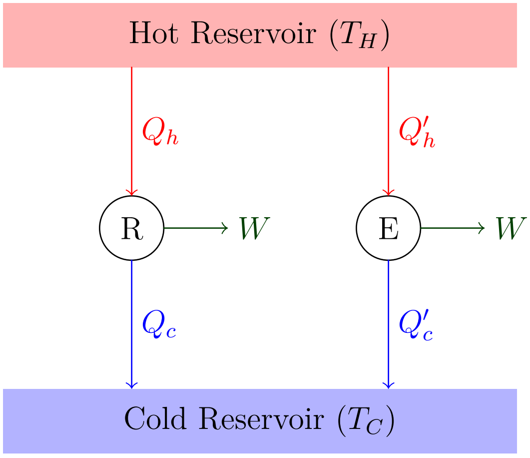

Fig. 1.3: A reversible and irreversible engine running between a hot and cold reservoir.

Both engines output work W but R takes in Q_h and outputs Q_c to T_C while E takes in

Q'_h and outputs Q'_c to T_C. For the sake of the proof, assume that $\eta_E > \eta_R$, i.e.

$$\frac{W}{Q'_h} > \frac{W}{Q_h}$$
$$Q_h > Q'_h. \tag{1.31}$$

Now, consider running R in reverse using the work done by E, so that R extracts heat Q_c from T_C and deposits heat Q_h into T_H.

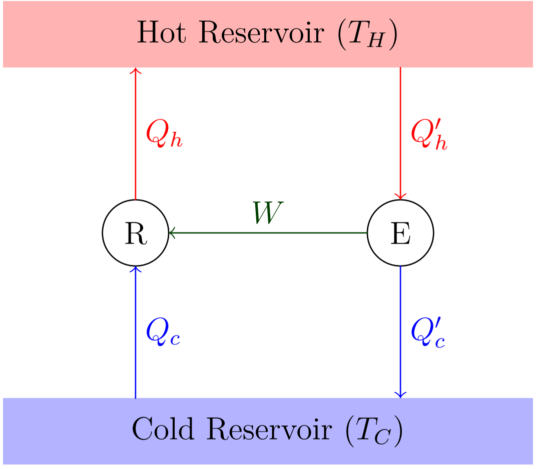

Fig. 1.4: A modified setup.

Analyzing the entire system at this state, the two engines extract heat $Q_c - Q'_c > 0$[6] from T_c and deposit heat $Q_h - Qh' > 0$ into T_h. Without any external work done on the system, this transfer of heat from a cold reservoir to a hot reservoir is a direct violation of the second law of thermodynamics, meaning that our initial assumption that $\eta_E > \eta_R$ must be false. Hence, there exists no engine more efficient than a reversible engine operating between two reservoirs.

If reversible engines are the most efficient engine between to temperature reservoirs, they must have the same efficiency, which is the maximum possible efficiency. Thus, the corollary mentioned earlier is also true. In real life applications, Carnot's theorem gives the upper bound on efficiency of a given heat engine, although most commercial engines don't come close to the ideal Carnot engine ★

1.2.5 ▶ Efficiency of a Carnot Engine

The Carnot Engine is a perfectly reversible engine operating between a hot and a cold temperature. As shown in 1.2.4, the efficiency of any such engine is the same for two reservoirs of given temperatures and is the upper bound for efficiency of an engine operating between the temperatures. A common way to construct the Carnot Engine is as follows (and as depicted in the diagram):

1. An isothermal expansion at hot temperature T_H

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[6] $Q_h = W + Q_c \Rightarrow$ e.q. 1.31 $\Rightarrow Q_c > Q'_c$

2. An adiabatic expansion
3. An isothermal compression at cold temperature T_C
4. An adiabatic compression

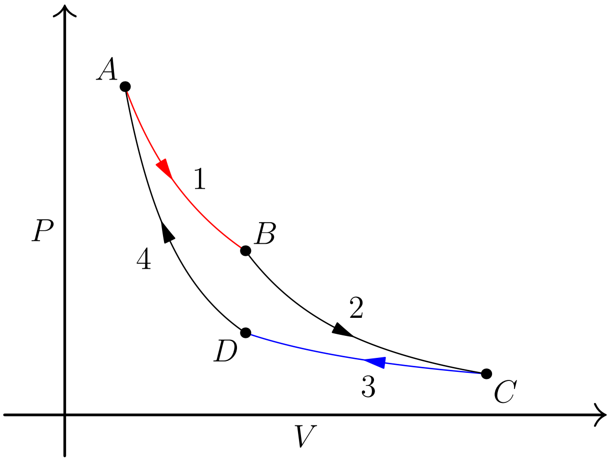

Fig. 1.5: Carnot Cycle, with two adiabatic and two isothermal processes. The hot temperature T_H is depicted in red while the cold temperature T_C is depicted in blue.

To find the efficiency of the engine, consider first the definition of engine efficiency, which is the ratio of the work output and the heat input. We can thus break each process down and see where heat is being inputted and where work is extracted (note that all of the following values are magnitudes).

1. Heat Q_H is inputted and work W_1 is extracted. Since $\Delta U = 0$, $Q_H = W_1$.[7]
2. No heat is exchanged but work W_2 is extracted.[8]
3. Heat Q_C is extracted and work W_3 is inputted. Since $\Delta U = 0$, $Q_C = W_3$.
4. No heat is exchanged but work W_4 is inputted.

By conservation of energy, the work output of the engine is $Q_H - Q_C$, meaning that the efficiency of the engine is

$$\epsilon_{carnot} = \frac{W_{out}}{Q_H} = \frac{Q_H - Q_C}{Q_H}. \tag{1.32}$$

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[7] Note here that the volume increases at a positive pressure, meaning that the quantity $W = -P\Delta V$ is negative, but the work that the gas does is positive. Why? It's because the quantity $-P\Delta V$ is the work done *on* the gas, not *by* the gas, i.e. it gives the work inputted into the gas, not the work extracted. An intuitive way of thinking of it is like this: it takes effort to compress a piston and decrease its volume (positive work input) and if released from a compressed state, the piston springs outwards (positive work output).

[8] Again, W_2 is a positive value despite there being an increase in volume.

Since $Q_H = W_1$, we can find Q_H by solving for the area under process 1:

$$\begin{aligned} Q_H = W_1 &= \int_{V_A}^{V_B} P dV \\ &= nRT_H \int_{V_A}^{V_B} \frac{dV}{V} \\ &= nRT_H \ln\left(\frac{V_B}{V_A}\right) \end{aligned} \tag{1.33}$$

where V_A and V_B are the volumes of the gas at state A and B, respectively. By the same logic,

$$Q_C = W_3 = \int_{V_D}^{V_C} P dV = nRT_C \ln\left(\frac{V_C}{V_D}\right) \tag{1.34}$$

Plugging eq. 2.33 and 2.34 into 2.32, we have

$$\epsilon_{carnot} = 1 - \frac{T_C}{T_H} \cdot \frac{\ln(V_B/V_A)}{\ln(V_C/V_D)} \tag{1.35}$$

From the derivation in 1.2.3 (specifically eq. 2.29), we can relate the volumes between the start and end points of the adiabatic processes

$$\begin{gathered} \frac{T_H}{T_C} = \left(\frac{V_C}{V_B}\right)^{\gamma-1} = \left(\frac{V_D}{V_A}\right)^{\gamma-1} \\ \frac{V_C}{V_D} = \frac{V_B}{V_A}. \end{gathered}$$

With this result, eq. 2.35 simply reduces to

$$\boxed{\epsilon_{carnot} = 1 - \frac{T_C}{T_H}}. \tag{1.36}$$

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APPENDIX **A**

Appendix

A.1 Useful Math References

A.1.1 Trigonometric Identities

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
1 + \cot^2 x &= \csc^2 x
\end{aligned}$$

$$\begin{aligned}
\sin(2x) &= 2 \sin x \cos x \\
\cos(2x) &= \cos^2 x - \sin^2 x \\
\tan(2x) &= \frac{2 \tan x}{1 - \tan^2 x}
\end{aligned}$$

$$\begin{aligned}
\sin \frac{x}{2} &= \pm\sqrt{\frac{1 - \cos x}{2}} \\
\cos \frac{x}{2} &= \pm\sqrt{\frac{1 + \cos x}{2}} \\
\tan \frac{x}{2} &= \frac{1 - \cos x}{\sin x} \\
&= \frac{\sin x}{1 + \cos x}
\end{aligned}$$

$$\begin{aligned}
\sin(x + y) &= \sin x \cos y + \cos x \sin y \\
\cos(x + y) &= \cos x \cos y - \sin x \sin y \\
\sin x + \sin y &= 2 \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right) \\
\cos x + \cos y &= 2 \cos\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right)
\end{aligned}$$

A.1.2 Taylor Series

$$
\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\
\frac{1}{1+x} &= 1 + x + x^2 + x^3 + \dots \{-1 < x < 1\}
\end{aligned}
$$

A.2 Helpful Proofs

A.2.1 Arc length and a Straight Line

Consider an arc subtended by a very small angle $d\theta$ in a circle of radius R.

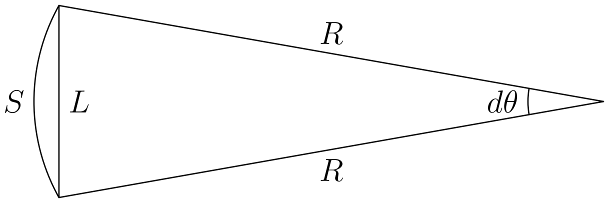

Fig. A.1

The distance S equals $R\theta$ while the distance L equals $2R\sin(d\theta/2)$. Taking the difference of these two lengths gives

$$
\begin{aligned}
S - L &= R\theta - 2R\sin\left(\frac{d\theta}{2}\right) \\
&= Rd\theta - 2R\left(\frac{d\theta}{2} - \frac{d\theta^3}{48} + \frac{d\theta^5}{3840} - \dots\right) \\
&\approx \frac{d\theta^3}{24} \qquad \text{(A.1)}
\end{aligned}
$$

which is third-order small and thus will always be smaller than $d\theta$, which is already essentially zero. Since the two segments only differ to the third order in $d\theta$, we consider them to be equal for small angles ★

A.2.2 Properties of Ellipses

There is some important ellipse notation that should be understood before approaching elliptical orbits. Let us start with a diagram of an ellipse (note that it's not to scale):

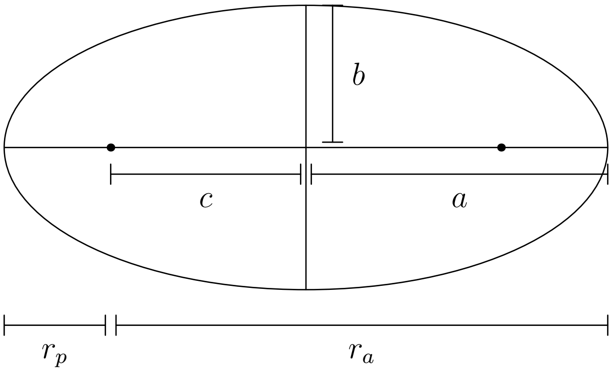

Fig. A.2

The distance a is the semi-major axis of the ellipse while the distance b is the semi-minor axis. The distance c is a representation of the eccentricity of the ellipse, i.e $c = \epsilon a$. If we place the star at the leftmost focus, r_p represents the perihelion distance while r_a represents the aphelion distance. From just these definitions, we can write

$$2a = r_a + r_p \tag{A.2}$$

$$2c = r_a - r_p \tag{A.3}$$

Next, by the definition of an ellipse, the distance from the top of the ellipse to each foci must be a. So, we can write

$$b^2 + c^2 = a^2 \tag{A.4}$$

Multiplying eq. A.4 through by 4 and substituting using A.2 and A.3 gives

$$4b^2 + r_a^2 - 2r_a r_p + r_p^2 = r_a^2 + 2r_a r_p + r_p^2$$

$$b = \sqrt{r_a r_p} \tag{A.5}$$

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A.2.3 Small Angle Approximations

For small angles, we often make a few approximations for trigonometric functions. $\sin(\theta)$ is approximated to θ, since writing out a taylor expansion yields

$$\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

Ignoring higher-order terms just yields $\sin\theta \approx \theta$. Doing the same for cosine yields

$$1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

Ignoring high-order terms gives $\cos\theta \approx 1 - \frac{\theta^2}{2}$ ★